Mathematics-I Lecture Material
Unit-I
MATRICES

for

B.Tech (ECE, CSE, AIML, CS, DS & IT)

I year – I semester

R18 Regulation

Unit-I Objectives:

1. Types of matrices and their properties.
2. Concept of a rank of the matrix and applying this concept to know the consistency and solving the system of linear equations.

Unit-I Outcomes:

1. Write the matrix representation of a set of linear equations and to analyse the solution of the system of equations

Compiled by

1. **Dr. A Nagaraju**, Professor of Mathematics
2. **Mrs. CH Harisha,** Associate professor of Mathematics
3. **Mrs. A Srilakshmi**, Assistant professor of Mathematics
4. **Mr. A Sreenivasulu**, Assistant professor of Mathematics
5. **Mrs. T Vasanthi Devi,** Assistant professor of Mathematics
6. **Mrs. K Santosh Rupa,** Assistant professor of Mathematics
7. **Mrs. K Swapna,** Assistant professor of Mathematics

Department of Humanities and Sciences

MALLA REDDY INSTITUTE OF TECHNOLOGY AND SCIENCE

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Maisammaguda, Dhulapally, Post via kompally, Secunderabad – 500 100

www.mrits.ac.in

UNIT-1

MATRICES

Matrix: A system of m*n numbers real (or) complex arranged in the form of an ordered set of 'm' rows,, each row consisting of an ordered set of 'n' numbers between [ ] (or) ( ) (or) || || is called a matrix of order m xn.

Eg: $\begin{bmatrix} a_{11}a_{12}\ldots\ldots\ldots a_{1n} \\ a_{21}a_{12}\ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}a_{m2}\ldots\ldots a_{mn} \end{bmatrix}$ $[a_{ij}]_{mxn}$ where $1\leq i\leq m$, $1\leq j\leq n$.

Some types of matrices:

1. **Square matrix:** A square matrix A of order nxn is sometimes called as a n- rowed matrix A (or) simply a square matrix of order n

   eg : $\begin{bmatrix} 11 \\ 22 \end{bmatrix}$is 2^{nd} order matrix

2. Rectangular matrix: A matrix which is not a square matrix is called a rectangular matrix,

$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$is a 2x3 matrix

3. Row matrix: A matrix of order 1xm is called a row matrix

eg: $[1\ 2\ 3]_{1x3}$

4. Column matrix: A matrix of order nx1 is called a column matrix

Eg: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3x1}$

5. **Unit matrix:** if A= $[a_{ij}]_{nxn}$ such that a_{ij} = 1 for i = j and a_{ij} = 0 for i≠j, then A is called a unit matrx.

   Eg:I_2 = $\begin{bmatrix} 10 \\ 01 \end{bmatrix}$ I_3=$\begin{bmatrix} 100 \\ 010 \\ 001 \end{bmatrix}$

6. **Zero matrix:** it A = $[a_{ij}]_{mxn}$ that a_{ij} = ∀i, j then A is called a zero matrix (or) null matrix

   Eg: O_{2x3}=$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7. **Diagonal elements in a matrix** A= $[a_{ij}]_{mxn}$, the elements a_{ij} of A for which i = j i.e. (a_{11}, a_{22}….a_{nn}) are called the diagonal elements of A

   Eg: A=$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$diagonal elements are 1,5,9

Note: the line along which the diagonal elements lie is called the principle diagonal of A

8. **Diagonal matrix:** A square matrix all of whose elements except those in leading diagonal are zero is called diagonal matrix.

   If d_1, d_2….. d_n are diagonal elements of a diagonal matrix, A, then A is written as A = diag (d_1, d_2…. dn)

   Eg : A = diag (3,1,-2)

9. **Scalar matrix:** A diagonal matrix whose leading diagonal elements are equal is called a scalar matrix. Eg : A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

10. **Equal matrices:** Two matrices A = [a_{ij}] and b= [b_{ij}] are said to be equal if and only if (i) A and B are of the same type (ii) $a_{ij} = b_{ij}$ for every i&j

11. **The transpose of a matrix:** The matrix obtained from any given matrix A, by inter changing its rows and columns is called the transpose of A. It is denoted by a^1 (or) a^T. If A = $[a_{ij}]_{mxn}$ then the transpose of A. is $A^1 = [b_{ij}]_{nxm}$, where $b_{ji} = a_{ij}$ Also $(a^1)^1 = A$

Note: A^1 and B^1 be the transposes of A and B repectively, then

(i) $(A^1)^1 = A$

(ii) $(A+B)^1 = A^1+B^1$

(iii) $(KA)^1 = KA^1$, K is a scalar

(iv) $(AB)^1= B^1A^1$

12. The conjugate of a matrix: The matrix obtained from any given matrix A, on replacing its elements by corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$

Note: if $\bar{A}$and $\bar{B}$ be the conjgates of A and B respectively then,

(i) $\overline{(\bar{A})} = A$

(ii) $\overline{(A+B)} = \bar{A}+\bar{B}$

(iii) $\overline{(KA)} = \bar{K}\bar{A}$, K is a any complex number

(iv) $\overline{(AB)}= \bar{B}\,\bar{A}$

Eg ; if A= $\begin{bmatrix} 2 & 3i & 2-5i \\ -i0 & 4 & i+3 \end{bmatrix}_{2x3}$ then$\bar{A} = \begin{bmatrix} 2 & -3i & 2+5i \\ i0 & -4 & i+3 \end{bmatrix}_{2x3}$

13. <u>The conjugate Transpose of a matrix</u>

The conjugate of the transpose of the matrix A is called the conjugate transpose of A and is denoted by A^θ Thus $A^\theta =(\bar{A^1})$ is the transpose of A^1 now a = $[a_{ij}]_{m\times n} \rightarrow A^\theta =[b_{ij}]n\times m$, where bij = $\bar{a}$ij i.e. the $(i,j)^{th}$ element of A^θ conjugate complex of the $(j, i)^{th}$ element of A

Eg: if A = $\begin{bmatrix} 5 & 3-i & -2i \\ 0 & 1+i & 4-i \end{bmatrix}$, then A^{θ} = $\begin{bmatrix} 5 & 0 \\ 3-i & 1-i \\ 2i & 4-i \end{bmatrix}_{3x2}$

14. Upper Triangular matrix: A square matrix all of whose elements below the leading diagonal are zero is called an Upper triangular matrix.

Eg; $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$

15. Lower triangular matrix; A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix

Eg: $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 6 \end{bmatrix}$

16. Symmetric matrix: A square matrix A =$[a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for every i and j Thus A is a symmetric matrix iff $A^T = A$

Eg: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ Is a symmetric matrix

17. Skew – Symmetric: A square matrix A = $[a_{ij}]$ is said to be skew – symmetric if $a_{ij} = -a_{ji}$ for every i and j. Thus A is a skew – symmetric iff $A = -A^T$

Eg : $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

18. **<u>Trace of A square matrix:</u>** Let A = $[a_{ij}]_{n\times n}$ the trace of the square matrix A is defined as $\sum_{i=1}^{n} a_{ii}$ and is denoted by 'tr A'

Thus trA = $\sum_{i=1}^{n} a_{ii} = a_{11}+a_{22}+ \ldots\ldots +a_{nn}$

<u>Properties:</u> If A and B are square matrices of order n and λ is any scalar, then

(i) tr (λ A) = λ tr A

(ii) tr (A+B) = trA + tr B

(iii) tr(AB) = tr(BA)

19. Idempotent matrix: If A is a square matrix such that $A^2 = A$ then 'A' is called idempotent matrix

20. Nilpotent matrix: If A is a square matrix such that $A^m=0$ where m is a +ve integer then A is called nilpotent matrix.

21. Involutory: If A is a square matrix such that $A^2 = I$ then A is called involuntary matrix.

22. Orthogonal matrix: A square matrix A is said to be orthogonal if $AA^1 = A^1A = I$.

23. Conjugate of a matrix:

If the elements of a matrix A are replaced by their conjugates, then the resulting matrix is defined as the conjugate of the given matrix. We denote it with $\overline{A}$

e.g If A=$\begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$ then $\overline{A}$=$\begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$

24. The transpose of the conjugate of a square matrix:

If A is a square matrix and its conjugate is $\overline{A}$, then the transpose of $\overline{A}$ is $(\overline{A})^T$.It can be easily seen that $(\overline{A})^T$=$\overline{A^T}$. It is denoted by A^θ

Note: If $A^\theta$and B^θ be the transposed conjugates of A and B respectively, then

i) $(A^\theta)^\theta = A$ ii) $(A \pm B)^\theta = A^\theta \pm B^\theta$ iii) $(KA)^\theta = \overline{K}A^\theta$ iv) $(AB)^\theta = B^\theta A^\theta$

25. Hermitian matrix:

A square matrix A such that $\overline{A}$=A^T (or) $(\overline{A})^T$=A is called a Hermitian matrix. Here $(\overline{A})^T$=A, Hence A is called Hermitian

e.g A=$\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ then $\overline{A}$=$\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ and A^θ=$\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

Note:

1) The element of the principal diagonal of a Hermitian matrix must be real
2) A hemitian matrix over the field of real numbers is nothing but a real symmetric.

26. Skew-Hermitian matrix

A square matrix A such that A^T=-$\overline{A}$ (or) $(\overline{A})^T$=-A is called a Skew-Hermitian matrix

e.g. Let A=$\begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$ then $\overline{A}$=$\begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$ $(\overline{A})^T$=$\begin{bmatrix} 3i & -2+i \\ 2-i & i \end{bmatrix}$

$\therefore (\overline{A})^T$=-A

A is skew-Hermitian matrix.

27. Unitary matrix:

A square matrix A such that $(\overline{A})^T$=A^{-1}. i.e $(\overline{A})^T$A=A$(\overline{A})^T$=I

If A^θA=I then A is called Unitary matrix

28. Rank of a Matrix

Let A be mxn matrix. If A is a null matrix, we define its rank to be 'o'. if A is a non-null matrix, we say that r is the rank of A if

I. Every (r+1)th order minor of A is 'o' (zero) &

II. At least one rth order minor of A which is not zero.

29. Normal Form:

Every mxn matrix of rank r can be reduced by a finite number of elementary transformations to the form $\begin{pmatrix} I_r 0 \\ 00 \end{pmatrix}$, where I_r is the r – rowed unit matrix.

Note: 1. If A is an mxn matrix of rank r, there exists non-singular matrices P and Q such that PAQ = $\begin{pmatrix} I_r 0 \\ 00 \end{pmatrix}$

30. Gauss – Jordan method

- **i.** suppose A is a non-singular matrix of order 'n' then we write A = I_n A
- **ii.** Now we apply elementary row-operations only to the matrix A and the pre-factor I_n of the R.H.S
- **iii.** We will do this till we get I_n = BA then obviously B is the inverse of A.

31. Non homogeneous working rule.

The system Ax = B is consistent if ρ(A) = ρ[A/B]

i). ρ(A) = ρ[A/B]= r < n(no. of unknowns).

Then there are infinite no of solutions.

ii). ρ(A) = ρ[A/B] = number of unknowns then the system will have unique solution.

iii). ρ(A) ≠ ρ[A/B] the system has no solution.

32. homogeneous working rule.

Working rule for finding the solutions of the equation Ax =0

(i). Rank of A = No. of unknowns i.e r = n the given system has zero solution.

(ii). Rank of A < No of unknowns (r<n) and No. of equations < No. of unknowns (m<n) then the system has infinite no. of solutions.

Short answer questions

1. show that A = $\begin{bmatrix} \cos\theta \ \sin\theta \\ -\sin\theta \cos\theta \end{bmatrix}$ is orthogonal.

Solution: Given A = $\begin{bmatrix} \cos\theta \sin\theta \\ -\sin\theta \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta - \sin\theta \\ \sin\theta \quad \cos\theta \end{bmatrix}$

Consider $A.A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 01 \end{bmatrix} = I$$

A is orthogonal matrix.

2. **Theorem:** Every square matrix can be expressed as the sum of a symmetric and skew – symmetric matrices in one and only way

Proof: let A be any square matrix. We can write

A= ½ (A+A^1)+ ½ (A-A^1)=P+Q (say).

Where P = ½ (A+A^1)

Q = ½ (A-A^1)

We have P^1 = {½ (A+A^1)}^{1} = ½ (A+A^1)^1 since [(KA)^1 = KA^1]

= ½ [A+(A^1)^1]= ½ [A+A^1]=P

P is symmetric matrix.

Now, Q^1 = [½ (A-A^1)]^1 = ½ (A-A^1)^1

= ½ [A^1-(A^1)^1] = ½ (A^1-A)

= - ½ (A-A^1)= -Q

Q is a skew – symmetric matrix.

Thus, square matrix = symmetric + skew – symmetric then to prove the sum is unique.

It possible, let A = R+S be another such representation of A where R is a symmetric one S is a skew – symmetric matrix.

R^1 = R and S^1 = -S

Now A^1 = (R+S)^1 = R^1+S^1 = R-S and

½ (A+A^1) = ½ (R+S+R-S) = R

½ (A-A^1) = ½ (R+S-R+S) = S

⇒R = P and S=Q

Thus, the representation is unique.

3. **Theorem2**: Prove that inverse of a non – singular symmetric matrix A = symmetric.

Proof: since A is non – singular symmetric matrix A^{-1} exists and A^T = A

Now, we have to prove that A^{-1} is symmetric we have $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ (by (1))

Since $(A^{-1})^T = A^{-1}$ therefore, A^{-1} is symmetric.

4. **Theorem3:** if A is a symmetric matrix, then prove that adj A is also symmetric

Proof: Since A is symmetric, we have $A^T = A$ … (1)

Now, we have $(adjA)^T = adj\ A^T$ [since adj $A^1 = (AdjA)^1$]

$= adj\ A$ [by (1)]

$(adjA)^T = adjA$ therefore, adjA is a symmetric matrix.

5. Express the matrix A as sum of symmetric and skew – symmetric matrices. Where

$A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 3 & -2 & 5 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

Matrix A can be written as $A = ½ (A+A^T) + ½ (A-A^T)$

⇨$P = ½ (A+A^T) = \frac{1}{2}\left\{\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & \frac{11}{2} \\ 0 & 7 & \frac{3}{2} \\ \frac{11}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

$Q= ½ (A-A^T)$

$= \frac{1}{2}\left\{\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & -2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}\right\}$

$$= \begin{bmatrix} 0 & -2 & \frac{1}{2} \\ 2 & 0 & -\frac{5}{2} \\ -\frac{1}{2} & \frac{5}{2} & 0 \end{bmatrix}$$

A = P+Q where 'P' is symmetric matrix

'Q' is skew-symmetric matrix.

6. find the adjoint and inverse of a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

Solution: Adjoint of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Where Aij are the cofactors of the elements of a_{ij}.

Cofactors $A_{ij} = (-1)^{i+j}\ M_{ij}$

Adjoint of A = $\begin{bmatrix} -4 & 11 & 4 \\ 4-11 & 6 \\ 8 & 8-8 \end{bmatrix}^T = \begin{bmatrix} -4 & 4 & 8 \\ 1-11 & 8 \\ 14 & 6-8 \end{bmatrix}$

$|A|$ = -4-2(-1) +3(14) = 40

$$A^{-1} = \frac{1}{|A|}(adjA)$$

$$= \frac{1}{40}\begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & 8 \end{bmatrix}$$

7. Solve the equations 3x+4y+5z = 18, 2x-y+8z =13 and 5x-2y+7z =20

Solution: The given equations in matrix form is AX = B

$$\begin{bmatrix} 3 & 4 & 5 \\ 2-1 & 8 \\ 5-2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

det A = 3(-7+16)-4(14-40) +5(-4+5) = 136

co-factor matrix is D = $\begin{bmatrix} (-7+16)-(14-40)(-4+5) \\ -(28+10)(21-25)-(-6-20) \\ (32+5)-(24-10)(-3-8) \end{bmatrix}$

D = $\begin{bmatrix} 9 & 26 & 1 \\ -38-4 & 26 \\ 37-14-11 \end{bmatrix}$

Adj A = D^T = $\begin{bmatrix} 9-38 & 37 \\ 26-4-14 \\ 126-11 \end{bmatrix}$

A^{-1} = 1/det A adj A = $\frac{1}{136}\begin{bmatrix} 9-38 & 37 \\ 26-4-14 \\ 126-11 \end{bmatrix}$

A x = B => x = A^{-1} B

$$= \frac{1}{136}\begin{bmatrix} 9-38 & 37 \\ 26-4-14 \\ 126-11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$= \frac{1}{136}\begin{bmatrix} 162-494-740 \\ 468-52-280 \\ 18+338-220 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{136}\begin{bmatrix} 408 \\ 136 \\ 136 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Solution is x =3, y=1, z=1.

8. **Theorem:** The Eigen values of a unitary matrix have absolute value l.

Proof: Let A be a square unitary matrix whose Eigen value is λ with corresponding eigen vector X

$\Rightarrow \quad AX = \lambda X \rightarrow (1)$

$\Rightarrow \quad \overline{AX} = \overline{\lambda X} \Rightarrow \overline{X}^T \overline{A}^T = \bar{\lambda} X^T \rightarrow (2)$

Since A is unitary, we have $(\overline{A})^T A = I \rightarrow (3)$

(1) and (2) given $\overline{X}^T \overline{A}^T \ AX = \lambda\bar{\lambda} X^T X$

i.e $\quad \overline{X}^T X = \lambda\bar{\lambda}\overline{X}^T X$

From (3) $\Rightarrow \overline{X}^T X(1 - \lambda\bar{\lambda}) = 0$

Since, $\overline{X}^T X \neq 0$,we must have $1 - \lambda\bar{\lambda} = 0$

$$\Rightarrow \lambda\bar{\lambda} = 1$$

Since, $|\lambda|$=$\overline{|\lambda|}$ We must have $\overline{|\lambda|}$=1.

9. **Theorem:** Prove that transpose of a unitary matrix is unitary.

Proof: let A be a unitary matrix

Then $A.A^\theta = A^\theta.A = I$

Where $A^\theta$the transposed is conjugated of A.

$\therefore \left(AA^\theta\right)^T = \left(A^\theta A\right)^T = (I)^T$

$\therefore \left(AA^\theta\right)^T = \left(A^\theta A\right)^T = (I)^T$

$\Rightarrow \left(A^\theta\right)^T A^T = A^T\left(A^\theta\right)^T = I$

$\Rightarrow (A^T)^\theta A^T = A^T(A^T)^\theta = I$

Hence A^T is a unitary matrix.

10. Find the eigen values of A=$\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

Solution: we have A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

So $\overline{A}$=$\begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$ and $A^T = \begin{bmatrix} 3i & -2+i \\ +2+i & -i \end{bmatrix}$

$\Rightarrow \overline{A}$=$-A^T$

Thus, A is a skew-Hermition matrix.

∴The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow |3i - \lambda \quad 2 + i \; -2 + i \quad -1 - \lambda| = 0$

$\Rightarrow \lambda^2 - 2i\lambda + 8 = 0$

$\Rightarrow \lambda = 4i, -2i$ Are the Eigen values of A

11. Find the eigen values of A=$\begin{bmatrix} 1/2i & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2i \end{bmatrix}$

Now $\overline{A}$=$\begin{bmatrix} -1/2i & 1/2\sqrt{3} \\ 1/2\sqrt{3} & -1/2i \end{bmatrix}$ and

$(\overline{A})^T$=$\begin{bmatrix} -1/2i & 1/2\sqrt{3} \\ 1/2\sqrt{3} & -1/2i \end{bmatrix}$

We can see that $\overline{A}^T . A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus, A is a unitary matrix

∴ The ch. equation is$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1/2i - \lambda & 1/2\sqrt{3} \\ 1/2\sqrt{3} & 1/2i - \lambda \end{vmatrix} = 0$$

Which gives $\lambda = 1/2\sqrt{3} + 1/2i$ and

$$\lambda = 1/2\sqrt{3} + 1/2i$$

Hence above λ values are Eigen values of A.

12. If A=$\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then show that A is Hermitian iA is skew-Hermitian.

Sol: Given A=$\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then

$\overline{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$ And $\overline{(A)}^T = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

$\therefore A = (\overline{A})^T$ Hence A is Hermitian matrix.

Let B= iA

i.e B=$\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$ then

$\overline{B} = \begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$ And $(\overline{B})^T = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix}$

$$(\overline{B})^T = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix}$$

$\therefore (\overline{B})^T$=-B

$\therefore$B= iA is a skew Hermitian matrix.

13. If A and B are Hermitian matrices, prove that AB-BA is a skew-Hermitian matrix.

Solution: Given A and B are Hermitian metrices

$\therefore (\overline{A})^T = A$ And $(\overline{B})^T = B$ ------------- (1)

Now $\overline{(AB-BA)}^T = (\overline{AB} - \overline{BA})^T$

$= (\overline{AB} - \overline{BA})^T$

$= (\overline{AB})^T - (\overline{BA})^T = (\overline{B})^T(\overline{A})^T - (\overline{A})^T(\overline{B})^T$

$= BA - AB$ (By (1))

$= -(AB - BA)$

Hence AB-BA is a skew- Hermitian matrix.

14. Show that A=$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if $a^2+b^2+c^2+d^2=1$

Solution: Given A=$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$

Then $\overline{A} = \begin{bmatrix} a-ic & -b-id \\ b-id & a+ic \end{bmatrix}$

Hence $A^\theta = (\overline{A})^T = \begin{pmatrix} a-ic & -b-id \\ b-id & a+ic \end{pmatrix}$

$A^\theta = (\overline{A})^T = \begin{pmatrix} a-ic & -b-id \\ b-id & a+ic \end{pmatrix}$

$\therefore AA^\theta = \begin{pmatrix} a=ic & -bid \\ b+id & a-ic \end{pmatrix}\begin{pmatrix} a-ic & b-id \\ -b-id & a+ic \end{pmatrix}$

=$\begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$

$\therefore AA^\theta = I$ if and only if $a^2+b^2+c^2+d^2=1$

15. Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew- Hermitian matrix.

Solution. Let A be any square matrix

Now $(A + A^\theta)^\theta = A^\theta + (A^\theta)^\theta$

$$= A^{\theta} + A$$

$\left(A + A^{\theta}\right)^{\theta} = A + A^{\theta} \Rightarrow A + A^{\theta}$ is a hemitian matrix.

$\Rightarrow \frac{1}{2}\left(A + A^{\theta}\right)$ is also a Hermitian matrix

Now $\left(A - A^{\theta}\right)^{\theta} = A^{\theta} - \left(A^{\theta}\right)^{\theta}$

$$= A^{\theta} - A = -(A - A)^{\theta}$$

Hence $A - A^{\theta}$ is a skew-Hermitian matrix

$\therefore \frac{1}{2}\left(A - A^{\theta}\right)$ Is also a skew –Hermitian matrix.

16. Given that A=$\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(1 - A)(1 + A)^{-1}$ is a unitary matrix.

Solution: we have $1 - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$ And

$$1 + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$\therefore (1 + A)^{-1} = \frac{1}{1-(4i^{-1})}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Let $B = (1 + A)(1 + A)^{-1}$

$$B = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Now $\overline{B} = \frac{1}{6}\begin{bmatrix} -4 & -2+4i \\ 2+4i & -4 \end{bmatrix}$ and $\overline{(B)}^{T} = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$$B\left(\overline{B}\right)^{T} = \frac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\left(\overline{B}\right)^T = B^{-1}$$

i.e B is unitary matrix.

$\therefore (1-A)(1+A)^{-1}$ Is a unitary matrix.

17. find the rank of the given matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 10 12 \end{bmatrix}$

solution: Given matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

→ det A = 1(48-40)-2(36-28) +3(30-28)

= 8-16+6 = -2 ≠ 0

We have minor of order 3 ≠ 0

P(A) =3.

18. find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ by reducing it to Echelon form.

solution: Given A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Applying row transformations on A.

A ~ $\begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

~$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 9 \\ 0 & 9 & 9 \end{bmatrix}$ $R_2 \rightarrow R_2 - 3R_1$

$R_3 \rightarrow R_3 - 2R_1$

~$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $R_3^1 \rightarrow R_3 - R_2$

~$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

This is the Echelon form of matrix A.

The rank of a matrix A.

= Number of non – zero rows =2

Long answer questions

19. For what values of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$ has rank '3'.

Solution: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

$$A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$$

Applying $R_2 \to 4R_2 - R_1$, $R_3 \to 4R_3 - kR_1$, $R_4 \to 4R_4 - 9R_1$

We get $A \sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$

Since Rank A = 3, det A =0

$$4 \begin{vmatrix} 0 - 1 - 1 \\ 8 - 4k8 + 3k \quad 8 - k \\ 04k + 273 \end{vmatrix} = 0$$

1[(8-4k)3]-1(8-4k) (4k+27)] = 0

(8-4k) (3-4k-27) = 0

(8-4k) (-24-4k) =0

(2-k) (6+k) =0

k =2 or k = -6

20. By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ into normal form, find its rank.

Solution: Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix}$ $R_3 \to R_3/-2$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ $R_3 \to R_3 + R_2$

$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ $c_2 - 2c_1$, c_3-3c_1, c_4-4c_1

$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix}$ $3c_3 - 2c_2$, $3c_4-5c_2$

$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $c_2/-3$, $c_4/18$

$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $c_4 \leftrightarrow c_3$

This is in normal form $[I_3\ 0]$

Hence Rank of A is '3'.

21. find the inverse of the matrix A using elementary operations.

Given $A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

We can write $A = I_3 A$

$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Applying $R_3 \rightarrow 2R_3 - R_2$, we get

$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} A$

Applying $R_1 \rightarrow R_1 - 3R_2$, we get

$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} A$

Applying $R_1 \rightarrow R_1 + 5R_3$, $R_2 \rightarrow R_2 - 3R_3/2$, we get

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 10 \\ 0 & 1/3 & -3 \\ 0 & 0 & 2 \end{bmatrix} A \Rightarrow I_3 = BA$

B is the inverse of A.

22. Discuss for what values of λ, μ the simultaneous equations $x+y+z = 6$, $x+2y+3z = 10$, $x+2y+\lambda z = \mu$ have

(i). no solution

(ii). A unique solution

(iii). An infinite number of solutions.

Soln: The matrix form of given system of Equations is

$$A\,x = \begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}6\\ 10\\ \mu\end{bmatrix} = B$$

The augmented matrix is $[A/B] = \begin{bmatrix}1 & 1 & 1 & 6\\ 1 & 2 & 3 & 10\\ 1 & 2 & \lambda & \mu\end{bmatrix}$

$$[A/B] \sim \begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 1 & 2 & 4\\ 1 & 1 & \lambda-1 & \mu-6\end{bmatrix} R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix}1 & 1 & 1 & 6\\ 0 & 1 & 2 & 4\\ 0 & 0 & \lambda-3 & \mu-10\end{bmatrix} R_3 \to R_3 - R_2$$

Case (i): let $\lambda \neq 3$ the rank of A = 3 and rank [AB] = 3

Here the no. of unknowns is '3'

Here ρ (A) =ρ(A/B)= No. of unknows

The system has unique solution if $\lambda \neq 3$ and for any value of 'μ'.

Case (ii). Suppose λ =3 and $\mu \neq 10$.

We have ρ(A)= 2 ρ(AB) = 3

The system have no solution.

Case (iii): let λ =3 and μ=10.

We have ρ(A)= 2 ρ(AB) = 2

Here ρ(A)= ρ(AB) $\neq$ No. of unknowns =3

The system has infinitely many solutions.

23. Show that the equations x+y+z = 4, 2x+5y-2z =3, x+7y-7z =5 are not consistent.

Solution: write given equations is of the form Ax = B

$$\text{i.e } \begin{bmatrix}1 & 1 & 1\\ 2 & 5 & -2\\ 1 & 7 & -7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}4\\ 3\\ 5\end{bmatrix}$$

consider Augment matrix i.e [A /B]

$$[A/B] = \begin{bmatrix}1 & 1 & 1 & 4\\ 2 & 5 & -2 & 3\\ 1 & 7 & -7 & 5\end{bmatrix}$$

Applying $R_2 \to R_2-2R_1$ and $R_3 \to R_3-R_1$, we get

$$[A/B] \sim \begin{bmatrix}1 & 1 & 1 & 4\\ 0 & 3 & -4 & -5\\ 0 & 6 & -8 & 1\end{bmatrix}$$

Applying $R_3 \to R_3-2R_1$, we get

$$[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

ρ (A) =2 and ρ(A/B) =3

The given system is inconsistent as

$$\rho(A) \neq \rho[A/B].$$

24. Show that the equations given below are consistent and hence solve them

x-3y-8z = -10, 3x+y-4z =0, 2x+5y+6z =3

Solution: matrix notation is $\begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 3 \end{bmatrix}$

Augmented matrix [A/B] is

$$[A/B] = \begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{bmatrix} R_2 \to R_2\text{-}3R_1$$

$R_3 \to R_3 -2R_1$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} R_2 \to 1/10\ R_2$$

$R_3 \to 1/11 R_3$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \sim R_3 \to R_3\text{-}R_2$$

This is the Echelon form of [AB]

P(A) =ρ(A/B) = 2<3 (no. of unknown)

The system has infinite number of soln.

The given system of equations is equal to

$$\begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ 3 \\ 0 \end{bmatrix}$$

x-3y-8z = -10

y+2z =3

Give arbitary value to z. i.e say z =k then y = 3-2k and x = ++2k

For different values of k, we have an infinite number of solutions.

25. Solve the system of equations x+3y-2z = 0, 2x-y+4z = 0, x-11y+14z = 0

Solution: We write the given system is Ax = 0

ie. $\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

A ~ $\begin{bmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & -14 & 16\end{bmatrix}$ $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - R_1$

~ $\begin{bmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & 0 & 0\end{bmatrix}$ $R_3 \rightarrow R_3 - 2R_2$

The Rank of the A = 2 ie. ρ(A)

No of unknowns is '3'

We have infinite No. of solution

Above matrix can we write as

x+3y-2z =0 -7y+8z =0, 0=0

say z = k then y=8/7k & x= -10/7 k

giving different values to k, we get infinite no. of values of x,y,z.

26. Show that the only real number λ for which the system x+2y+3z = λx, 3x+3y+z = λz, has non-zero solution is 6 and solve them.

Solution: Above system can we expressed as Ax = 0

ie. $\begin{bmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$

given system of equations possess a non –zero solution i.e ρ(A) < no. of unknowns.

For this we must have det A = 0

$\begin{bmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{bmatrix} = 0$

$\begin{bmatrix}6-\lambda & 6-\lambda & 6-\lambda\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{bmatrix} = 0$ $R_1 \rightarrow R_1+R_2+R_3$

(6-λ) $\begin{bmatrix}1 & 1 & 1\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{bmatrix} = 0$

(6-λ) $\begin{bmatrix}1 & 0 & 0\\ 3 & -2-\lambda & -1\\ 2 & 1 & -1-\lambda\end{bmatrix} = 0$ $c_2 \rightarrow c_2 - c_1$

$c_3 \rightarrow c_3 - c_1$

(6-λ) [(-2- λ) (-1- λ) +1] =0

$(6-\lambda)(\lambda^2+3\lambda+3) = 0$

$\lambda = 6$ only real values.

When $\lambda = 6$, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \to 5R_2+3R_1,\ R_3 \to 5R_3+2R_1$$

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3+R_2$$

$-5x+2y+3z = 0$ and $-19y+19z = 0$

$y = z$

Say $z = k$, $y = k$ and $x = k$.

Solution is $x = y = z = k$.

27. Solve the system of eqns $3x+y-z = 3$, $2x-8y+z = -5$, $x-2y+9z = 8$ using Gauss elimination method.

Solution: The argumented Matrix is $[A\ B] = \begin{bmatrix} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{bmatrix}$

Performing $R_2 \to R_2 - {}^2/_3 R_1$

$R_3 \to R_3 - {}^1/_3 R_1$, we get

$$[A\ B] \sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -\frac{26}{3} & \frac{5}{3} & -7 \\ 0 & \frac{-7}{3} & \frac{28}{8} & 7 \end{bmatrix}$$

$$[A\ B] \sim \begin{bmatrix} 3 & 1 & -1 & 3 \\ 0 & -\frac{26}{3} & \frac{5}{3} & -7 \\ 0 & 0 & \frac{693}{78} & \frac{231}{26} \end{bmatrix}$$

$R_3 \to R_3 - {}^7/_{26} R_2$

From above we get

$3x+y-z = 3$

$-26/3\ y + 5/3\ x = -7$

$693/78\ x = 231/26$

$x = 1, y = 1, z = 1$

28.

Use Gauss-Seidel iteration method to solve the system of equations

$$\mathbf{10x + y + z = 12}$$

$$\mathbf{2x + 10y + z = 13}$$

$$\mathbf{2x + 2y + 10z = 14}.$$

Sol: The given system of equations are $10x + y + z = 12$

$$2x + 10y + z = 13$$

$$2x + 2y + 10z = 14$$

The given system is diagonally dominant and rewrite the given system of equations are

First Iteration :

$$x_1^{(1)} = \frac{1}{10}[12 - y^{(0)} - z^{(0)}] \ldots\ldots\ldots\ldots(1)$$

$$y_1^{(1)} = \frac{1}{10}[13 - 2x_1^1 - z^{(0)}] \ldots\ldots\ldots\ldots(2)$$

$$z_1^{(1)} = \frac{1}{10}[14 - 2x_1^{(1)} - 2y_1^{(1)}] \ldots\ldots\ldots(3)$$

we start iteration by taking $y^{(0)}=0$, $z^{(0)}=0$ in (1) we get

$$x_1^{(1)} = \frac{1}{10}[12 - 0 - 0] = 1.2$$

$$x_1^{(1)} = 1.2$$

Putting $x_1^{(1)} = 1.2$, $z^{(0)}=0$ in (2) we get

$$y_1^{(1)} = \frac{1}{10}[13 - 2(1.2) - 0] = 1.06$$

$$y_1^{(1)} = 1.06$$

Putting $x_1^{(1)} = 1.2$, $y_1^{(1)} = 1.06$ in (3) ,we get

$$z_1^{(1)} = \frac{1}{10}[14 - 2(1.2) - 2(1.06)] = 0.95$$

$$z_1^{(1)} = 0.95$$

Hence the first iteration values of x, y, z are $x_1^{(1)} = 1.2$, $y_1^{(1)} = 1.06$, $z_1^{(1)} = 0.95$.

Second iteration :

$$x_1^{(2)} = \frac{1}{10}[12 - y_1^{(1)} - z_1^{(1)}] \ldots\ldots\ldots\ldots(4)$$

$$y_1^{(2)} = \frac{1}{10}[13 - 2x_1^{(2)} - z_1^{(1)}] \ldots \ldots (5)$$

$$z_1^{(2)} = \frac{1}{10}[14 - 2x_1^{(2)} - 2y_1^{(2)}] \ldots\ldots\ldots (6)$$

Putting $y_1^{(1)}, z_1^{(1)}$ values in equation (4) ,then we get

$$x_1^{(2)} = 0.999$$

Putting $x_1^{(2)}$, $z_1^{(1)}$ values in equation (5) ,then we get

$$y_1^{(2)} = 1.005$$

Putting $x_1^{(2)}$, $y_1^{(2)}$ values in equation (6) , the we get

$$z_1^{(2)} = 0.999$$

Hence the second iteration values of x,y, z are $x_1^{(2)} = 0.999$, $y_1^{(2)} = 1.005$, $z_1^{(2)} = 0.999$.

Again taking $x_1^{(2)} = 0.999$, $y_1^{(2)} = 1.005$, $z_1^{(2)} = 0.999$ as the initial values ,we get

$$x_1^{(3)} = \frac{1}{10}[12 - 1.005 - 0.999] = 0.999 = 1.00$$

$$y_1^{(3)} = \frac{1}{10}[13 - 2.0 - 0.999] = 1.00$$

$$z_1^{(3)} = \frac{1}{10}[14 - 2 - 2] = 1.00$$

Hence the 3rd Iteration values os x,y,z are $x_1^{(3)} = 1.00$, $y_1^{(3)} = 1.00$, $z_1^{(3)} = 1.00$.

Similarly ,we find the 4th iteration values are $x_1^{(4)} = 1.00$,$y_1^{(4)} = 1.00$,$z_1^{(4)} = 1.00$

We tabulate the results as follows

Variable	1st Iteration	2nd Iteration	3rd Iteration	4th Iteration
x	1.20	0.999	1.00	1.00
y	1.06	1.005	1.00	1.00
z	0.95	0.999	1.00	1.00

Hence the solution of the given system of equations is $x = 1, y = 1, z = 1$